

Newtonian mechanics of neutron superfluid in elastic star crust.

Brandon Carter & Elie Chachoua

Observatoire de Paris, 92195 Meudon, France.

26 January, 2006

Abstract To account for pulsar frequency glitches, it is necessary to use a neutron star crust model allowing not only for neutron superfluidity but also for elastic solidity. These features have been treated separately in previous treatments of crust matter, but are combined here in a unified treatment that is based on the use of a Lagrangian master function, so that the coherence the system is ensured by the relevant Noether identities. As well as the model obtained directly from the variation principle, the same master function can provide other conservative alternatives, allowing in particular for the effect of perfect vortex pinning. It is also shown how such models can be generalised to allow for dissipative effects, including that of imperfect pinning, meaning vortex drag or creep.

1 Introduction

Almost immediately after the discovery of pulsars, and their identification as rotating neutron stars, it was recognised that, to account for their observed frequency glitches, a model of the purely fluid type such as is commonly sufficient in stellar structure theory would not suffice, and that an adequate “basic picture” [1] would require the use of a more elaborate kind of model allowing for the elasticity of the solid crust. In order to do this, a suitable category of elastic solid models was developed in which, as well as allowing for the effect of General Relativity (which, in the crust layers, is significant but not of overwhelming importance) the main innovation [2] was to use a treatment based on elastic perturbations not with respect to a fully relaxed local state (such as will usually be available in the context of terrestrial engineering) but with respect to a local state that is only conditionally relaxed with respect to perturbations that preserve

its density (which in a neutron star may be extremely high compared with the density at which the solid would be fully relaxed).

The motivation for the effort needed to construct global neutron star models [3] of such a purely elastic type was however diminished to some extent after it became clear that solidity alone could not explain the high rate at which quite large glitches were actually observed to occur. To account for this, it was generally recognised to be necessary to invoke a mechanism involving angular momentum transfer from a rather more rapidly rotating superfluid neutron constituent that can flow without resistance through the ionic crust material. To describe such a phenomenon at a macroscopic (local but not global) level, various multiconstituent fluid models have been developed over the years, both in a Newtonian framework [4] and also (not so much for the minor improvement in accuracy that it provides in principle, as because it is for many purposes actually easier to work in practise!) in a relativistic framework [5, 6, 7].

Work on the application, at the global stellar structure level, of such multiconstituent fluid models has by now been developed, both in a Newtonian framework [8], and in a relativistic framework [9], to such an extent that it now seems worthwhile to make the further step of incorporating the effects of solidity, without which the occurrence of the actual glitch phenomenon would not be possible. In order to provide what is missing [10] in a purely fluid description of such phenomena, the present article shows how a suitably chosen master function Λ can be used to set up a variational or more general Newtonian model of a neutron superfluid flowing within an elastic solid background, using a formalism that synthesises the separate descriptions of solidity [11] and multiconstituent fluidity [12, 13] that are already available in a form adapted for this purpose. An analogous synthesis in a relativistic framework is under preparation elsewhere [14].

2 Newtonian action formulation

2.1 Dynamic variables and background fields

Our purpose in this Section is to set up a multipurpose Newtonian variational formalism that combines the ones that have recently been developed on one hand for the treatment of a multiconstituent fluid [12, 13] and on the other hand for a non conducting elastic solid [11]. The treatment in the present section does not allow for dissipative effects, but is nevertheless designed in such a way as to facilitate their subsequent inclusion in the Section 5, using the approach that has recently been developed for the multiconstituent fluid case [15].

Let us start by recapitulating some basic considerations for the generic case, in a spacetime with arbitrarily chosen coordinates x^μ ($\mu = 0, 1, 2, 3$), of an action density scalar, Λ say, whose role is to act as a Lagrangian master function governing the evolution of a set of active or “live” dynamical fields, but whose complete specification also involves a set of given – passive or “dead” – background fields.

The most general infinitesimal Eulerian (i.e. fixed point) variation of such a Lagrangian will be decomposable as the sum of two contributions in the form

$$\delta\Lambda = \delta^\heartsuit\Lambda + \delta^\ddagger\Lambda \quad (1)$$

in which $\delta^\heartsuit\Lambda$ is the realisable part attributable to a physically possible alteration of the configuration of the “live” dynamical fields, while $\delta^\ddagger\Lambda$ is a virtual part arising from mathematically conceivable but (in the context under consideration) physically forbidden variations of the “dead” background fields that have been fixed in advance. In a typical special relativistic application the only relevant “dead” field might just be the Minkowski background metric. What we shall be concerned with here however is the more complicated Newtonian case in which, as explained in recent preceding work [11], the necessary set of “dead” fields include the rank-3 spacemetric $\gamma^{\mu\nu}$ and the associated time covector t_μ (which are restricted to be respectively spacelike and timelike in the sense of satisfying $t_\mu\gamma^{\mu\nu} = 0$) together with a (Galilean gauge fixing) ether vector e^μ . As well as these indispensable uniform background fields, the set of the “dead” background fields that are fixed in advance will be taken here, in the first instance, to include the generically non uniform gravitational potential ϕ , so that the complete background variation contribution will be given by an expression of the form

$$\delta^\ddagger\Lambda = \frac{\partial\Lambda}{\partial\gamma^{\mu\nu}}\delta\gamma^{\mu\nu} + \frac{\partial\Lambda}{\partial t_\mu}\delta t_\mu + \frac{\partial\Lambda}{\partial e^\nu}\delta e^\nu + \frac{\partial\Lambda}{\partial\phi}\delta\phi. \quad (2)$$

For more general purposes, the scalar ϕ would need to be promoted from being a given background to the status of a “live” dynamical field, which means that the last term in (2) would be transferred to the “live” variation contribution $\delta^\heartsuit\Lambda$, which in that case [13] would also include a term allowing for the dependence of Λ not just on ϕ but also on its gradient with components $\phi_{,\mu}$.

It is to be remarked that the construction of many of the relevant quantities will involve the rank 3 Euclidean space metric that is induced by the choice of the ether gauge vector e^μ according to the specifications

$$\eta_{\mu\nu}e^\nu = 0, \quad \eta_{\mu\nu}\gamma^{\nu\rho} = \delta_\mu^\rho - e^\rho t_\mu = \eta_\mu^\rho. \quad (3)$$

In the more specialised work on the fluid case [13] this uniform space metric was denoted by $\gamma_{\mu\nu}$, but the latter will be used here for the variable space metric that is to be defined below. As noted in our preceding work [11], the induced variation of the uniform space metric (3) will be expressible (using round brackets to indicate index symmetrisation) by the formula

$$\delta\eta_{\mu\nu} = -\eta_{\mu\rho}\eta_{\nu\sigma}\delta\gamma^{\rho\sigma} - 2t_{(\mu}\eta_{\nu)\rho}\delta e^\rho, \quad (4)$$

in which the last term shows the effect just of an infinitesimal Galilean transformation, under which $\delta\gamma^{\rho\sigma}$ would simply vanish.

The particular kind [12] of background variation in which we shall be interested is one that is simply generated by the action of a displacement field ξ^ν say, so that

the ensuing field variations will be given just by the negatives of the corresponding Lie derivatives. For the background fields that are uniform (in the sense of having vanishing covariant derivatives) the resulting variations will be given just by

$$\delta\gamma^{\mu\nu} = 2\gamma^{\rho(\mu}\nabla_{\rho}\xi^{\nu)} \quad \delta t_{\mu} = -t_{\nu}\nabla_{\mu}\xi^{\nu}, \quad \delta e^{\nu} = e^{\mu}\nabla_{\mu}\xi^{\nu}, \quad (5)$$

while for the gravitational potential we shall simply have

$$\delta\phi = -\xi^{\nu}\nabla_{\nu}\phi. \quad (6)$$

In so far as the “live” contribution is concerned, a further subdivision arises in the cases such as those of interest here, which are characterised by a constrained variation principle, meaning one whereby the “on shell” evolution condition is that of invariance of the relevant action integral with respect to a compactly supported perturbation of the dynamical fields that is not entirely arbitrary but constrained by an appropriate admissibility condition. In the applications under consideration, the relevant admissibility condition will be interpretable as the requirement for the change to represent a “natural” variation of the same given physical system, whereas more general changes would represent a replacement of the system by a slightly different system within the same category. In such a case the “live” field contribution will be decomposable (though not necessarily in a unique manner) as a sum of the form

$$\delta^{\heartsuit}\Lambda = \delta^{\natural}\Lambda + \delta^{\sharp}\Lambda \quad (7)$$

in which $\delta^{\natural}\Lambda$ denotes a part that would be inadmissible for the purpose of application of the variational principle, whereas $\delta^{\sharp}\Lambda$ is a “natural” variation that would be allowed for this purpose. This means that for any (unperturbed) configuration that is “on shell” – in the sense of satisfying the dynamical evolution equations provided by the constrained variation principle – a generic admissible variation must satisfy condition

$$\delta^{\sharp}\Lambda \cong 0, \quad (8)$$

using the symbol \cong to indicate equivalence modulo the addition of a divergence (which, by Green’s theorem, will give no contribution to the action integral from a variation that is compactly supported).

In the kind of medium with which we are concerned, the constrained variables will be a set of current 4-vectors n_{x}^{μ} with “chemical” index label x (which, in the kind of astrophysical applications we have in mind, might include the separate neutronic and protonic contributions to the conserved total baryonic current) while the unconstrained fields will consist of a triplet of scalar fields q^{A} (A = 1, 2, 3) that are interpretable [11] as local coordinates on a 3-dimensional material base manifold, and from which, together with their gradient components $q^{\text{A}}_{,\mu}$, an associated particle number (which might count ionic nuclei or crystalline solid lattice points in the relevant microscopic substructure) density current, n_{I}^{ν} say, can be constructed in the manner prescribed below. The

corresponding “live” part of the generic action variation will thus be expressible in the form

$$\delta^\heartsuit \Lambda = \Sigma_x \pi_\mu^x \delta n_x^\mu + \mathcal{P}_A^\nu \delta q_{,\nu}^A + \frac{\partial \Lambda}{\partial q^A} \delta q^A, \quad (9)$$

in which, for each separate current, the partial derivative $\pi_\mu^x = \partial \Lambda / \partial n_x^\mu$ is interpretable as the corresponding 4-momentum per particle.

It is to be observed that the partial derivative $\mathcal{P}_A^\nu = \partial \Lambda / \partial q_{,\nu}^A$ has the noteworthy property of being tensorial not just with respect to transformations of the space time coordinates x^ν but also with respect to transformations $q^A \rightarrow \tilde{q}^A$ of the material coordinates, q^A whose effect will simply be given by

$$q^A \mapsto \tilde{q}^A \quad \Rightarrow \quad \mathcal{P}_A^\nu \mapsto \mathcal{P}_B^\nu \frac{\partial q^B}{\partial \tilde{q}^A}, \quad (10)$$

This good behaviour contrasts with the comportment of the remaining partial derivative, $\partial \Lambda / \partial q^A$, for which the corresponding transformation has the non-tensorial form

$$\frac{\partial \Lambda}{\partial q^A} \mapsto \frac{\partial \Lambda}{\partial q^B} \frac{\partial q^B}{\partial \tilde{q}^A} + \mathcal{P}_C^\nu \tilde{q}_{,\nu}^B \frac{\partial^2 q^C}{\partial \tilde{q}^B \partial \tilde{q}^A}. \quad (11)$$

For each separate current n_x^μ , the admissible variations (as generated [12] by world-line displacements) will be specifiable by a corresponding displacement vector field, ξ_x^μ say, by the formula

$$\delta^\sharp n_x^\mu = -\vec{\xi}_x \mathcal{L} n_x^\mu - n_x^\mu \nabla_\nu \xi_x^\nu, \quad (12)$$

in which the Lie derivative is just the commutator

$$\vec{\xi}_x \mathcal{L} n_x^\mu = \xi_x^\nu \nabla_\nu n_x^\mu - n_x^\nu \nabla_\nu \xi_x^\mu. \quad (13)$$

The variations of the material base coordinate fields q^A will similarly be expressible in terms of their own displacement vector field ξ^μ , by an expression of the corresponding form

$$\delta^\sharp q^A = -q_{,\mu}^A \xi^\mu, \quad (14)$$

but in this case it is evident that no restriction is entailed, so that without loss of generality we can take $\delta^\sharp q^A = 0$, which means that the inadmissible part (if any) of a “live” variation will reduce simply to

$$\delta^\sharp \Lambda = \Sigma_x \pi_\nu^x \delta^\sharp n_x^\nu. \quad (15)$$

It can be seen (using integration by parts) that, in terms of the displacement fields introduced by (12) and (14), the generic admissible variation for the current carrying medium will be expressible –modulo a variationally irrelevant divergence term– in the form

$$\delta^\sharp \Lambda \cong -f_\nu^S \xi^\nu - \Sigma_x f_\nu^x \xi_x^\nu, \quad (16)$$

in which, for each value of x , the covectorial coefficient f_ν^x will be interpretable as the non-gravitational force density acting on the corresponding separate constituent, while the extra coefficient f_ν^S is the supplementary force density due, as discussed below, to stratification or solid elasticity, that acts on the underlying ionic (crystalline or glass like) lattice structure.

If we postulate that the system should obey the constrained variational principle that is expressible as the imposition of the “on shell” condition (8), then it is evident from (16) that the ensuing dynamical field equations will be expressible simply as the separate vanishing of each one of these force densities, i.e. as the requirement that we should have $f_\nu^S = 0$ and (for each value of x) $f_\nu^x = 0$. We shall however be mainly concerned with other possibilities, in particular with cases in which some of the constituents, labelled $x=c$ and $x=\emptyset$, are “confined” in the sense of being subject to a convection condition of the form

$$n_c^\nu q_{,\nu}^A = 0, \quad n_\emptyset^\nu q_{,\nu}^A = 0, \quad (17)$$

which means that they have to move with the underlying ionic flow. When the application of the variation principle is subject to the corresponding convective constraints

$$\xi_c^\nu = \xi_\emptyset^\nu = \xi^\nu, \quad (18)$$

the ensuing system of dynamical field equations will no longer entail the separate vanishing of f_ν^S , f_ν^\emptyset and f_ν^c , but only of the amalgamated ionic force density f_ν^I that is defined as their sum,

$$f_\nu^I = f_\nu^S + f_\nu^c + f_\nu^\emptyset, \quad (19)$$

which is interpretable as the net force density acting on the integrated system consisting of the ions in combination with the convected constituent.

2.2 Noether identity for Newtonian covariance

As in the preceding work [12], let us now consider the Noether type identity that is obtained (regardless of whether or not the variational field equations are satisfied) by taking the variations of all the relevant fields, “live” as well as “dead” to be given by the action of the same displacement field ξ^ν whose effect on the background fields was given by (5) and (6), so that the resulting effect on the Lagrangian scalar will be given just by the corresponding displacement variation

$$\delta\Lambda = -\xi^\nu \nabla_\nu \Lambda. \quad (20)$$

For the unconstrained fields q^A the effect of such a displacement will be given, according to (14), simply by

$$\delta q^A = -\vec{\xi} \mathcal{L} q^A = \delta^\sharp q^A. \quad (21)$$

On the other hand, for the constrained variables n_x^ν , it can be seen by setting ξ_x^ν to ξ^ν in (12) that we shall have

$$\delta n_x^\nu = -\vec{\xi} \mathcal{L} n_x^\nu = \delta^\sharp n_x^\nu + \delta^\sharp n_x^\nu, \quad (22)$$

in which the extra “unnatural” (variationally inadmissible) contribution will be given by

$$\delta^\# \mathbf{n}_x^\nu = \mathbf{n}_x^\nu \nabla_\mu \xi^\mu. \quad (23)$$

It can thereby be seen from (1) using (16) and (20) that, as an identity which will hold, modulo a divergence, for any dynamical field configuration whether or not it is “on shell”, we shall have a relation of the form

$$\xi^\nu (f_\nu^S + \Sigma_x f_\nu^x - \nabla_\nu \Lambda) \cong \delta^\# \Lambda + \delta^\dagger \Lambda. \quad (24)$$

We now evaluate the terms on the right by substitution of (23) in (15) and by substitution of the formulae (5) and (6) in (2) for an arbitrary displacement field ξ^ν . The identity (24) can thereby be rewritten more explicitly in the form

$$\xi^\nu (f_\nu^S + \Sigma_x f_\nu^x - \rho \nabla_\nu \phi) \cong -T^\mu{}_\nu \nabla_\mu \xi^\nu \cong \xi^\nu \nabla_\mu T^\mu{}_\nu, \quad (25)$$

in which the effective (passive) gravitational mass density can be read out simply as

$$\rho = -\frac{\partial \Lambda}{\partial \phi}, \quad (26)$$

while the corresponding (gravitationally passive) geometric stress momentum energy density tensor can be read out as

$$T^\mu{}_\nu = (\Lambda - \Sigma_x \pi_x^\rho \mathbf{n}_x^\rho) \delta_\nu^\mu - 2\gamma^{\mu\rho} \frac{\partial \Lambda}{\partial \gamma^{\rho\nu}} + \frac{\partial \Lambda}{\partial t_\mu} t_\nu - e^\mu \frac{\partial \Lambda}{\partial e^\nu}. \quad (27)$$

Since the arbitrary field ξ^ν can be taken to be non zero only in the immediate neighbourhood of any chosen point, the Noether identity (25) implies that, at each point, this stress energy tensor will satisfy a divergence identity of the simple form

$$\nabla_\mu T^\mu{}_\nu = f_\nu^S + \Sigma_x f_\nu^x - \rho \nabla_\nu \phi. \quad (28)$$

In the “on shell” case for which the variational evolution equations are satisfied, the non gravitational force density contributions on the right of (28) will drop out, leaving just the final term on the right which represents the gravitational force density which will always be present since the gravitational potential ϕ is treated as a background field.

The tensor $T^\mu{}_\nu$ can be employed in the usual way for construction of the energy flux vector U^ν and the energy density U according to the prescriptions

$$U^\mu = -T^\mu{}_\nu e^\nu, \quad U = t_\mu U^\mu, \quad (29)$$

while the corresponding space momentum flux $\Pi^{\mu\nu}$ and space momentum density Π^μ will be given by the prescriptions

$$\Pi^{\mu\nu} = T^\mu{}_\sigma \gamma^{\sigma\nu} \quad \Pi^\nu = t_\mu \Pi^{\mu\nu}, \quad (30)$$

which can then be used to determine a pressure tensor $P^{\mu\nu}$ that is specified by the decomposition

$$\Pi^{\mu\nu} = P^{\mu\nu} + e^\mu \Pi^\nu. \quad (31)$$

(It is to be remarked, as a minor caveat, that the systematic notation scheme previously developed for cases in which all the constituents were of purely fluid type [13] deviated slightly from the – even more systematic – scheme used here, not only by using $\gamma_{\mu\nu}$ for what is denoted here by $\eta_{\mu\nu}$, but also by using P^μ_ν just as an abbreviation for $T^\mu_{\text{int}\sigma}$.) It can be seen from the Noetherian construction (29) that the pressure tensor that is specified in this way will automatically be symmetric and strictly spacelike,

$$P^{\mu\nu} = P^{\nu\mu}, \quad P^{\mu\nu} t_\nu = 0, \quad (32)$$

since it will be given explicitly by

$$P^{\mu\nu} = (\Lambda - \Sigma_x \pi^\chi_\sigma n^\sigma_x) \gamma^{\mu\nu} - 2 \gamma^{\mu\rho} \frac{\partial \Lambda}{\partial \gamma^{\rho\sigma}} \gamma^{\sigma\nu}. \quad (33)$$

It can also be seen that the 3-momentum density will contain no contribution from any part of the Lagrangian that is independent of the Galilean frame, as is the case for the purely internal action contribution Λ_{int} that will be discussed below: we shall simply have

$$\Pi^\nu = - \frac{\partial \Lambda}{\partial e^\sigma} \gamma^{\sigma\nu} \quad \Rightarrow \quad \Pi^\nu_{\text{int}} = 0. \quad (34)$$

The corresponding formula for the scalar energy density is

$$U = \Sigma_x \pi^\chi_\sigma n^\sigma_x - \Lambda - t_\sigma \frac{\partial \Lambda}{\partial t_\sigma} + e^\sigma \frac{\partial \Lambda}{\partial e^\sigma}. \quad (35)$$

2.3 Canonical formulation

If, instead of working it out in terms of force densities modulo a divergence as in (24) and (25), we insert the displacement contributions (21) and (22) to the “live” variation (9) directly in (1) we get an identity in which the coefficients of the locally adjustable fields ξ^μ and $\nabla_\mu \xi^\nu$ must vanish separately. The former of these conditions reduces to a triviality, but the latter provides a relation showing that the geometrically defined stress energy tensor (28) can be rewritten in the equivalent canonical form

$$T^\mu_\nu = \Lambda \delta^\mu_\nu + \Sigma_x (\pi^\chi_\nu n^\mu_x - \pi^\chi_\rho n^\rho_x \delta^\mu_\nu) - \mathcal{P}^\mu_\nu, \quad (36)$$

in which the stress contribution at the end is given in terms of the coefficient introduced in (10) by

$$\mathcal{P}^\mu_\nu = \mathcal{P}^\mu_A q^A_{,\nu}. \quad (37)$$

The “live” variation formula (9) also provides what is needed for the derivation of the corresponding expressions for the force densities introduced in (16) which for the fluid constituents will have the form that is already familiar from the preceding work [12]. In terms of the corresponding generalised vorticity 2-forms defined, using square brackets for index antisymmetrisation, by

$$\varpi_{\mu\nu}^x = 2\nabla_{[\mu}\pi_{\nu]}^x, \quad (38)$$

these non-gravitational current force densities will be given by

$$f_\nu^x = n_x^\mu \varpi_{\mu\nu}^x + \pi_\nu^x \nabla_\mu n_x^\mu, \quad (39)$$

while the non-gravitational force density acting on the underlying atomic structure of the medium will be given by the prescription

$$f_\mu^S = \frac{\delta\Lambda}{\delta q^A} q_{,\mu}^A, \quad (40)$$

in which the Eulerian derivative is given by a prescription of the usual form

$$\frac{\delta\Lambda}{\delta q^A} = \frac{\partial\Lambda}{\partial q^A} - \nabla_\nu \mathcal{P}_A^\nu, \quad \mathcal{P}_A^\nu = \frac{\partial\Lambda}{\partial q_{,\nu}^A}. \quad (41)$$

It is important to notice that the non-tensorial base coordinate transformation property (11) of the first term in this formula will be cancelled by that of the second, so that the effect on the Eulerian derivative of a change of the material base coordinates will be given simply by

$$q^A \mapsto \tilde{q}^A \quad \Rightarrow \quad \frac{\delta\Lambda}{\delta q^A} \mapsto \frac{\delta\Lambda}{\delta \tilde{q}^B} \frac{\partial \tilde{q}^B}{\partial q^A}, \quad (42)$$

which shows that the (stratification and solid elasticity) force density (40) is invariant with respect to such transformations, an important property that is not so obvious from its detailed expression

$$f_\mu^S = \frac{\partial\Lambda}{\partial q^A} q_{,\mu}^A + \mathcal{P}_A^\nu \nabla_\mu q_{,\nu}^A - \nabla_\nu (\mathcal{P}_A^\nu q_{,\mu}^A), \quad (43)$$

in which only the final term is separately invariant.

It is important to observe that, regardless of what – variational or other – dynamical field equations may be imposed, the force density (43) will never be able to do any work on the medium since, as an obvious consequence of (40) it must automatically satisfy the further identity

$$f_\nu^S u^\nu = 0, \quad (44)$$

where u^μ is the unit 4-velocity of the medium as specified by the defining conditions

$$q_{,\nu}^A u^\nu = 0, \quad t_\nu u^\nu = 1. \quad (45)$$

If the variation principle (8) is imposed, so that – as remarked above – by (16) the separate force densities f_μ^S and f_μ^x will all have to vanish, then it evidently follows from the Noether identity (28) that the divergence condition

$$\nabla_\mu T^\mu{}_\nu = -\rho \nabla_\nu \phi, \quad (46)$$

must hold. This condition is interpretable as an energy-momentum balance equation that must be satisfied whenever the system is not subject to any non-gravitational external forces. So long as we are concerned only with a system that is isolated in this sense, so that the energy-momentum balance condition (46) is satisfied, it follows from (28) that, even for a more general model admitting the presence of dissipative or other internal forces, these must be specified in such a way as to satisfy the total force balance condition

$$f_\nu^S + \Sigma_x f_\nu^x = 0, \quad (47)$$

in order for the model to be self consistent. More particularly, in view of (44) it can be seen that the fluid force densities by themselves will have to satisfy the condition

$$u^\mu \Sigma_x f_\mu^x = 0, \quad (48)$$

expressing energy balance in the local rest frame of the underlying medium.

3 Application to superfluid neutron conduction in convective solid

3.1 Implementation of the confinement constraint

In the appendix of the third part [15] of the preceding series of articles on multiconstituent fluid models in a Newtonian framework, an account was provided of the kind of application that provided the main motivation for this work, namely a model in which the chemical variable x runs just three values, which are $x=f$ and $x=c$ say, labelling a “confined” baryon current n_c^ν and a “free” baryon current n_f^ν , together with a third value, $x=\emptyset$ say, labelling an entropy current $s^\nu = s u_\emptyset^\nu$, which (because of the relatively low temperatures involved) will play a relatively minor role in the pulsar applications we have in mind. The new feature in the present treatment is allowance for the solidity that (again because of the relatively low temperatures involved) will characterise the “confined” constituent, whose flow will therefore be constrained to be aligned with the flow vector u^ν of the solid structure as introduced above, so that it will have the form

$$n_c^\nu = n_c u^\nu. \quad (49)$$

As in the previous treatment [15] we shall restrict our attention here to the case in which the (in any case relatively unimportant) entropy current is convected with the

solid structure, so that the thermal rest frame will be identifiable with that of the solid, $u_\emptyset^\nu = u^\nu$, which means that the entropy current will be constrained to have a form analogue to (49), namely

$$s^\nu = s u^\nu. \quad (50)$$

The part of the baryon current that is confined according to (49) (meaning that has to be comoving with the underlying atomic or crystal structure of the medium) will add up with the “free” part to give a conserved total baryon current

$$n_b^\nu = n_c^\nu + n_f^\nu, \quad \nabla_\nu n_b^\nu = 0. \quad (51)$$

It can be seen to follow from this current conservation requirement that, in terms of the “free” current vorticity 2-form $\varpi_{\mu\nu}^f$, the energy conservation identity (48) will reduce to the form

$$u^\mu \varpi_{\mu\nu}^f n_f^\nu = (\mathcal{E}^c - \mathcal{E}^f) \nabla_\nu n_f^\nu - \Theta \nabla_\nu s^\nu, \quad (52)$$

in which the relevant rest-frame particle energies, \mathcal{E}^c and \mathcal{E}^f , and the temperature, $\mathcal{E}^\emptyset = \Theta$ say, are given in terms of the corresponding 4-momenta, as introduced in (9), by the definitions

$$\mathcal{E}^c = -u^\mu \pi_\mu^c, \quad \mathcal{E}^f = -u^\mu \pi_\mu^f, \quad \Theta = -u^\mu \pi_\mu^\emptyset. \quad (53)$$

It is to be noticed that these definitions (53) are not affected by the freedom to adjust the specifications of the confined particle 4-momentum covector π_μ^c and the thermal 4-momentum covector π_μ^\emptyset due to the constraints (49) and (50) on the corresponding currents n_c^ν and s^ν , of which the former can be seen to have the form (17) which evidently means that the specification of $q_{,\nu}^A$ in (9) will also be ambiguous. These ambiguities will however need to be resolved in order for us to be able to proceed in a well defined manner. To obtain a formalism that matches smoothly with what has already been set up for the fluid limit [15], we need to remove the ambiguity in π_μ^c by the imposition on π_μ^\emptyset and $q_{,\nu}^A$ of appropriate conditions, which can be taken to be that the former should only have no space component, which means that will be given in terms of the temperature Θ by

$$\pi_\mu^\emptyset = -\Theta t_\mu, \quad (54)$$

while the latter should have only space components, meaning that it will satisfy the condition

$$\mathcal{P}_A^\nu t_\nu = 0. \quad (55)$$

The stress energy tensor (36) will thereby be obtained in the form

$$T^\mu_\nu = (\Lambda + \Theta s - \pi_\rho^f n_f^\rho - \pi_\rho^c n_c^\rho) \delta^\mu_\nu - \Theta s^\mu t_\nu + \pi_\nu^f n_f^\mu + \pi_\nu^c n_c^\mu - \mathcal{P}_A^\mu q_{,\nu}^A. \quad (56)$$

(Instead of the convention (55), an obvious alternative ansatz would be to use the analogue of (54) to the effect that π^c_μ should have no space component, so that it would take the form $\pi^c_\mu = -\mathcal{E}^c t_\mu$. Such an alternative choice would be adequate for the specialised purpose of matching the formalism used for the non-conducting solid limit case [11], but not for broader objective of consistency with the formalism developed for the multiconstituent fluid limit case [12, 13, 15].)

It is computationally convenient and physically natural to take the material base space to be endowed with a measure form say that is specifiable in terms of antisymmetric components, n_{IABC} say, that are fixed in the sense of depending only on the q^A , and that will determine a corresponding scalar space-time field n_I by the determinant formula

$$n_I^2 = \frac{1}{3!} n_{IABC} n_{IDEF} \gamma^{AD} \gamma^{BE} \gamma^{CF}, \quad (57)$$

in which γ^{AB} denotes the induced base space metric

$$\gamma^{AB} = \gamma^{\mu\nu} q^A_{,\mu} q^B_{,\nu}, \quad (58)$$

whose components will be time dependent, unlike those of the measure n_{IABC} . For purposes of physical interpretation it will usually be convenient to take this measure to represent the number density of ionic nuclei, so that the confined baryon number density $n_c = t_\nu n_c^\nu$ will be given by

$$n_c = A_c n_I, \quad (59)$$

where A_c is the atomic number, meaning the number of confined baryons (protons plus confined neutrons) per nucleus. It is to be observed that, whereas the number current n_c^ν will fail to be conserved in the chemical equilibrium case characterised by (76), the formalism is such that the corresponding ionic number current

$$n_I^\nu = n_I u^\nu \quad (60)$$

must automatically satisfy the conservation law

$$\nabla_\nu n_I^\nu = 0 \quad (61)$$

as a mathematical identity.

3.2 Chemical gauge adjustments

For particular physical applications, the specification of the part of the baryon current that is to be treated as “confined” will be to some extent dependent on the timescales of the processes under consideration as compared with the lifetimes for quantum tunnelling through the confining barriers containing the nuclei. For relatively rapid processes the fraction of the relevant quantum states that should be considered to be confined may be subject to an increase, so the corresponding current, \tilde{n}_c^ν say, of baryons that are

effectively confined will be somewhat larger than the value, n_c^ν say, that would be appropriate for processes occurring over longer timescales. It is therefore of interest to consider the effect of such a chemical basis adjustment, as given by a transformation of the form

$$\tilde{n}_c^\nu = (1 + \epsilon)n_c^\nu, \quad \tilde{n}_f^\nu = n_f^\nu - \epsilon n_c^\nu, \quad (62)$$

for some suitably specified dimensionless adjustment parameter ϵ . So long as ϵ is just a constant, it is immediately apparent that such a transformation will leave the “free” 4-momentum covector invariant, meaning that we shall have

$$\pi_\nu^f = \tilde{\pi}_\nu^f, \quad (63)$$

and it is also easy to see that in terms of the new variables the stress energy tensor(56) will also be expressible in the same canonical form as before,

$$T_\nu^\mu = \tilde{T}_\nu^\mu. \quad (64)$$

As in the fluid case [16] this covariance property of the canonical stress energy tensor is not restricted to transformations for which ϵ is constant, but can be seen to hold whenever ϵ is specified as a function just of the atomic number $A_c = n_c/n_I$, and of the material position coordinates q^A . Since it can be seen from (57) that the variation of the ionic number will have a “live” (fixed background) part expressible in the form

$$\delta^\heartsuit n_I = n_I \gamma_{AB} q_{,\mu}^B \gamma^{\mu\nu} \delta q_{,\nu}^A + \frac{\partial n_I}{\partial q^A} \delta q^A, \quad (65)$$

it follows that the corresponding variation of the adjustment parameter will have the form

$$\delta^\heartsuit \epsilon = \frac{\partial \epsilon}{\partial A_c} \left(\frac{\delta^\heartsuit n_c}{n_I} - A_c \gamma_{AB} q_{,\mu}^B \gamma^{\mu\nu} \delta q_{,\nu}^A \right) + \left(\frac{\partial \epsilon}{\partial q^A} - \frac{\partial \epsilon}{\partial A_c} \frac{A_c}{n_I} \frac{\partial n_I}{\partial q^A} \right) \delta q^A. \quad (66)$$

Since we have $\delta^\heartsuit n_c = t_\nu \delta n_c^\nu$ it follows that unlike the free particle momentum, which is subject to the chemical invariance condition (63), the confined particle momentum covector will undergo a non-trivial chemical adjustment given by

$$\pi_\nu^c = \tilde{\pi}_\nu^c + (\tilde{\pi}_\mu^c - \tilde{\pi}_\mu^f) \left(\epsilon \delta_\nu^\mu + A_c \frac{\partial \epsilon}{\partial A_c} u^\mu t_\nu \right), \quad (67)$$

while for the coefficient in the final term of (64) we shall also obtain a non-trivial adjustment given by

$$\mathcal{P}_A^\nu = \tilde{\mathcal{P}}_A^\nu - (\tilde{\pi}_\mu^c - \tilde{\pi}_\mu^f) n_c^\mu A_c \frac{\partial \epsilon}{\partial A_c} \gamma_{AB} q_{,\rho}^B \gamma^{\rho\nu}. \quad (68)$$

It can however be seen that the extra terms proportional to $\partial \epsilon / \partial A_c$ will cancel out in the final expression (56) for the canonical stress energy tensor, whose chemical gauge invariance property (64) is thus confirmed.

3.3 Vortex pinning and chemical equilibrium

The chemical gauge invariance of the “free” (though not the “confined”) 4-momentum covector is essential for the modelisation of superfluidity, whose effect on a mesoscopic scale (large compared with the microscopic lattice spacing but small compared with intervortex separation) will be expressed by the condition that this covector should have the form of a gradient, $\pi_{\mu}^f = \hbar \nabla_{\mu} \varphi$ of a periodic quantum phase angle φ , with the corollary that the corresponding chemically invariant (and even Milne invariant [12]) “free” vorticity tensor $\varpi_{\mu\nu}^f$ will also vanish at the mesoscopic level. At the macroscopic level, on a scale large compared with the separation between vortices, this vorticity will have a large scale average value that will not vanish, but to be compatible with a mesoscopic fibration by 2 dimensional vortex lines it must still be algebraically restricted by the degeneracy condition

$$\varpi_{\mu[\nu}^f \varpi_{\rho\sigma]}^f = 0. \quad (69)$$

This condition will automatically be satisfied by the field equations that will result from the relevant variational principle, which will require that the vanishing perturbation condition (8) should be satisfied, not for independent variations of all four of the displacement fields ξ_f^{μ} , ξ_c^{μ} , ξ_{\emptyset}^{μ} , ξ^{μ} , which in view of the constraint (49) would lead to overdetermination, but just for variations satisfying the corresponding restraint (18). This requirement entails the vanishing, not of all four of the relevant force densities f_{μ}^f , f_{μ}^c , f_{μ}^{\emptyset} , f_{μ}^S given by the prescriptions (39) and (43), but only of the free force density f_{μ}^f and of the amalgamated force density f_{μ}^I given by (19) that acts on the part that is convected with the ionic lattice.

According to the general formula (39), the vanishing of the free force density f_{μ}^f entails not only the separate conservation condition

$$\nabla_{\nu} n_f^{\nu} = 0, \quad (70)$$

for the free part n_f^{μ} of the current density but also a dynamic equation of the familiar form

$$\varpi_{\mu\nu}^f n_f^{\nu} = 0, \quad (71)$$

which using (38) can be seen to imply

$$n_f \mathcal{L} \varpi^f = 0, \quad (72)$$

which is interpretable as meaning that the vortex lines are “frozen” into the free fluid in the sense of being dragged along with it. It follows, according to (69) that there will be no dissipation in the sense that the entropy will be conserved,

$$\nabla_{\nu} s^{\nu} = 0. \quad (73)$$

Instead of postulating the strict application of the variation principle, we can obtain a non-dissipative model of another physically useful kind by postulating that the vortices

are “pinned” in the sense of being frozen, not into the fluid but into the atomic lattice structure, which means replacing (71) by a dynamical equation of the analogous form

$$\varpi_{\mu\nu}^f u^\nu = 0, \quad (74)$$

which unlike (71) is unaffected by the chemical base transformations considered in Subsection 3.2.

Whichever of the alternative possibilities (71) or (74) is adopted, the degeneracy condition (69) will evidently be satisfied, and more particularly the left hand side of the identity (52) will vanish, which means that in order to satisfy the entropy conservation condition (73) the model must be such as to satisfy the condition

$$(\mathcal{E}^c - \mathcal{E}^f) \nabla_\nu n_f^\nu = 0. \quad (75)$$

In the strictly variational case this requirement will obviously be implemented by the separate conservation condition (70), which will be physically realistic in the context of high frequency oscillations. However in applications to slow long term variations it will be more realistic to use a model based on the alternative possibility, namely that of chemical equilibrium in the solid’s rest frame, as given by the relation

$$\mathcal{E}^c = \mathcal{E}^f, \quad (76)$$

which, like (74) can be seen to be unaffected by the chemical base transformations considered in Subsection 3.2.

Thus, depending on how we choose between the alternatives (71) or (74), and between the alternatives (70) or (76), we can use the same Lagrangian master function Λ for the specification of 4 different kinds of non-dissipative model, which are categorisable as unpinned with separate conservation or chemical equilibrium, and pinned with separate conservation or chemical equilibrium.

4 Specification of the action

4.1 External action contribution

To apply the procedure described above, one needs to prescribe a (primary) equation of state specifying the functional dependence of the Lagrangian master function Λ on the relevant independent variables.

In accordance with the general principles described in the preceding work [11], the relevant Lagrangian will be decomposable in the form

$$\Lambda = \Lambda_{\text{ext}} + \Lambda_{\text{int}}, \quad (77)$$

in which the gauge dependent external part is given in terms of a fixed baryon mass parameter m by an expression of the familiar form

$$\Lambda_{\text{ext}} = \frac{1}{2} m (n_c v_c^2 + n_f v_f^2) - m n_b \phi. \quad (78)$$

The confined particle 3-velocity v_c^ν and the superfluid 3-velocity v_f^ν are defined here in terms of the corresponding 4-velocities u_c^ν and u_f^ν in the usual way, by setting

$$n_c^\nu = n_c u_c^\nu = n_c(e^\nu + v_c^\nu), \quad n_f^\nu = n_f u_f^\nu = n_f(e^\nu + v_f^\nu), \quad (79)$$

and the squares in (78) are defined by

$$v_c^2 = \eta_{\mu\nu} v_c^\mu v_c^\nu, \quad v_f^2 = \eta_{\mu\nu} v_f^\mu v_f^\nu, \quad (80)$$

where $\eta_{\mu\nu}$ is the uniform rank-3 space metric defined for the Galilean frame characterised by the ether frame vector e^ν according to the specifications (3). This means that the kinetic action contribution is deemed here to be independent a priori of the material coordinates q^A and their gradients, to which it will however be related “on shell” by the application a posteriori of the constraint (49) to the effect that the crust frame should coincide with that of the confined particles, i.e.

$$u^\nu = u_c^\nu, \quad v^\nu = v_c^\nu. \quad (81)$$

This approach differs from the treatment used for the non-conducting solid limit case [11], in which the constraint (81) was imposed in advance, but it leads to results that are entirely equivalent on shell. The strategy used here is designed so as to satisfy the condition (55) to the effect that

$$t_\mu \mathcal{P}^\mu_\nu = 0, \quad (82)$$

which has the advantage of ensuring that (as shown in Subsection 4.5) it will be fully consistent with the formalism that has been developed [12, 13, 15] for the multiconstituent fluid limit case.

In the formulation used here, the external action provides no contribution at all to the extra stress term \mathcal{P}^μ_ν so – as in the multiconstituent fluid case [13] – the external contribution to the stress energy tensor will be given by the simple formula

$$T^\mu_{\text{ext}\nu} = n_f^\mu p_\nu^f + n_c^\mu p_\nu^c - m \phi n_b^\mu t_\nu = n_f^\mu \pi_{\text{ext}\nu}^f + n_c^\mu \pi_{\text{ext}\nu}^c, \quad (83)$$

in which free and confined kinematic momentum covectors are given respectively by

$$p_\nu^f = m(\eta_{\nu\mu} v_f^\mu - \frac{1}{2} v_f^2 t_\nu), \quad p_\nu^c = m(\eta_{\nu\mu} v_c^\mu - \frac{1}{2} v_c^2 t_\nu), \quad (84)$$

and the corresponding non-local external momentum covectors are given by

$$\pi_{\text{ext}\nu}^f = p_\nu^f - m \phi t_\nu, \quad \pi_{\text{ext}\nu}^c = p_\nu^c - m \phi t_\nu. \quad (85)$$

Since, according to (34) the internal part will not contribute, the 3-momentum density (30) will consist just of the external part, which evidently provides the (chemically invariant) result

$$H^\nu = m(n_f v_f^\nu + n_c v_c^\nu) = m \eta_\mu^\nu n_b^\mu. \quad (86)$$

4.2 Internal action contribution

It is to be remarked that the Galileian frame independent difference $\mathbf{v}_f^\nu - \mathbf{v}^\nu$ determines a corresponding – purely spacelike – relative current vector,

$$\mathbf{n}_\perp^\mu = \mathbf{n}_f(\mathbf{v}_f^\nu - \mathbf{v}^\nu) = \mathbf{n}_f^\nu - \mathbf{n}_f \mathbf{u}^\nu, \quad \mathbf{n}_\perp^\mu t_\mu = 0, \quad (87)$$

which as well as being unaffected by changes of the Galileian frame is also unaffected by chemical base transformations of the form (62), which will simply give $\tilde{\mathbf{n}}_\perp^\nu = \mathbf{n}_\perp^\nu$.

The internal contribution Λ_{int} in (77) has to be independent of the choice of the Galileian ether frame vector \mathbf{e}^μ . This means that at a given material base location, as specified by the fields q^A , this internal contribution will depend only on the scalars $s = s^\nu t_\nu$, $\mathbf{n}_f = \mathbf{n}_f^\nu t_\nu$ and $\mathbf{n}_c = \mathbf{n}_c^\nu t_\nu$ and on the material projections

$$\mathbf{n}^A = \mathbf{n}_\perp^\mu q_{,\mu}^A = \mathbf{n}_f^\mu q_{,\mu}^A, \quad \gamma^{AB} = \gamma^{\mu\nu} q_{,\mu}^A q_{,\nu}^B, \quad (88)$$

so that its generic variation will be given by

$$\delta \Lambda_{\text{int}} = \frac{\partial \Lambda_{\text{int}}}{\partial s} \delta s + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_c} \delta \mathbf{n}_c + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_f} \delta \mathbf{n}_f + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}^A} \delta \mathbf{n}^A + \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{AB}} \delta \gamma^{AB} + \frac{\partial \Lambda_{\text{int}}}{\partial q^A} \delta q^A. \quad (89)$$

This provides an associated “convective” variation [17] (in which, with respect to appropriately dragged coordinates, both q^A and δq_ν^A are held constant) of the form

$$\delta_c \Lambda_{\text{int}} = \frac{\partial \Lambda_{\text{int}}}{\partial s} \delta_c s + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_c} \delta_c \mathbf{n}_c + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_f} \delta_c \mathbf{n}_f + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_\perp^\nu} \delta_c \mathbf{n}_\perp^\nu + \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} \delta_c \gamma^{\mu\nu}, \quad (90)$$

in terms of tensorial coefficients defined by

$$\frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_\perp^\nu} = \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}^A} q_{,\nu}^A, \quad \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} = \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{AB}} q_{,\nu}^A q_{,\mu}^B. \quad (91)$$

For the purpose of the present analysis, what we need is the corresponding “live” variation, as carried out at a fixed position in a fixed background, for which we obtain

$$\delta^\circ \Lambda_{\text{int}} = \frac{\partial \Lambda_{\text{int}}}{\partial s} \delta s + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_c} \delta \mathbf{n}_c + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}_f} \delta \mathbf{n}_f + \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}^A} \delta (q_{,\nu}^A \mathbf{n}_f^\nu) + 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{AB}} \gamma^{\mu\nu} q_{,\mu}^B \delta q_{,\nu}^A + \frac{\partial \Lambda_{\text{int}}}{\partial q^A} \delta q^A. \quad (92)$$

We now use the constraint $\mathbf{u}^\nu \delta q_{,\nu}^A = -q_{,\nu}^A \delta \mathbf{u}^\nu$ to recombine the terms in such a way as to obtain a coefficient \mathcal{P}_A^μ satisfying the condition (55) in an expression of the standard form

$$\delta^\circ \Lambda_{\text{int}} = -\Theta \delta s + \chi_\nu^c \delta \mathbf{n}_c^\nu + \chi_\nu^f \delta \mathbf{n}_f^\nu + \mathcal{P}_A^\nu \delta q_{,\nu}^A + \frac{\partial \Lambda_{\text{int}}}{\partial q^A} \delta q^A, \quad (93)$$

It can be seen that the required (purely spacelike) value of \mathcal{P}_A^μ will be given by

$$\mathcal{P}_A^\mu = \frac{\partial \Lambda_{\text{int}}}{\partial \mathbf{n}^A} \mathbf{n}_\perp^\mu + 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{AB}} q_{,\nu}^B \gamma^{\mu\nu}, \quad (94)$$

while the corresponding expressions for the internal contributions to the 4-momentum covectors will be given by

$$\Theta = -\frac{\partial \Lambda_{\text{int}}}{\partial s}, \quad \chi_{\nu}^c = \frac{\partial \Lambda_{\text{int}}}{\partial n_c} t_{\nu} - \frac{n_f}{n_c} \frac{\partial \Lambda_{\text{int}}}{\partial n_{\perp}^{\nu}}, \quad \chi_{\nu}^f = \frac{\partial \Lambda_{\text{int}}}{\partial n_f} t_{\nu} + \frac{\partial \Lambda_{\text{int}}}{\partial n_{\perp}^{\nu}}, \quad (95)$$

which implies that we shall have

$$\frac{\partial \Lambda_{\text{int}}}{\partial n_c} = u^{\nu} \chi_{\nu}^c, \quad \frac{\partial \Lambda_{\text{int}}}{\partial n_f} = u^{\nu} \chi_{\nu}^f. \quad (96)$$

It follows, according to the canonical formula (36) that the internal contribution to the stress energy tensor will be given by

$$T_{\text{int}\nu}^{\mu} = -\Theta s^{\mu} t_{\nu} + \chi_{\nu}^c n_c^{\mu} + \chi_{\nu}^f n_f^{\mu} + \Psi \delta_{\nu}^{\mu} - \mathcal{P}_{\nu}^{\mu}, \quad (97)$$

with

$$\mathcal{P}_{\nu}^{\mu} = \frac{\partial \Lambda_{\text{int}}}{\partial n_{\perp}^{\nu}} n_{\perp}^{\mu} + 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\nu\rho}} \gamma^{\rho\mu}, \quad (98)$$

and with the generalised pressure scalar Ψ given, as in the multiconstituent fluid case, by

$$\Psi = \Lambda_{\text{int}} + \Theta s - \chi_{\nu}^f n_f^{\nu} - \chi_{\nu}^c n_c^{\nu}. \quad (99)$$

The identity (32) ensures the symmetric and strictly spacelike nature of the ensuing pressure tensor, as given in accordance with (31) by

$$P_{\text{int}}^{\mu\nu} = T_{\text{int}\sigma}^{\mu} \gamma^{\sigma\nu}, \quad (100)$$

which must also be symmetric. It will be obtainable according to the prescription

$$P_{\text{int}}^{\mu\nu} = (s\Theta - (n_f \chi_{\sigma}^f + n_c \chi_{\sigma}^c) u^{\sigma} - \chi_{\sigma}^f n_{\perp}^{\sigma}) \gamma^{\mu\nu} - S^{\mu\nu}, \quad S^{\mu\nu} = \gamma^{\mu\rho} \gamma^{\nu\sigma} S_{\rho\sigma}, \quad (101)$$

in which the stress contribution $S_{\mu\nu}$ is given by an expression of the same form as in the non conducting solid case [11], namely

$$S_{\mu\nu} = 2 \frac{\partial \Lambda_{\text{int}}}{\partial \gamma^{\mu\nu}} - \Lambda_{\text{int}} \gamma_{\mu\nu} = 2 n_I \frac{\partial(\Lambda_{\text{int}}/n_I)}{\partial \gamma^{\mu\nu}} = 2 n_I \frac{\partial(\Lambda_{\text{int}}/n_I)}{\partial \gamma_{AB}} q_{,\mu}^A q_{,\nu}^B. \quad (102)$$

This space projected part can be combined with the associated time projected part, namely the comoving energy current defined by

$$U_{\text{int}}^{c\mu} = -T_{\text{int}\nu}^{\mu} u^{\nu}, \quad (103)$$

to give back the entire internal stress-energy tensor in the form

$$T_{\text{int}\nu}^{\mu} = P_{\text{int}}^{\mu\rho} \gamma_{\rho\nu} - U_{\text{int}}^{c\mu} t_{\nu}, \quad (104)$$

in which $\gamma_{\mu\nu}$ is the time dependent covariant metric that is given by

$$\gamma_{\mu\nu} = \gamma_{AB} g_{,\mu}^A g_{,\nu}^B, \quad (105)$$

where the covariant base space metric γ_{AB} is such that

$$\gamma_{AB} \gamma^{BC} = \delta_A^C. \quad (106)$$

Using (45) it can be seen that $\gamma_{\mu\nu}$ is rank 3 :

$$\gamma_{\mu\nu} u^\nu = 0. \quad (107)$$

One can thus interpret this covariant metric as the rank 3 Euclidian metric that is obtained by substituing the ether frame vector defined in (3) by the solid's reference frame u^μ . As a consequence $\gamma_{\mu\nu}$ will be such that

$$\gamma_{\mu\rho} \gamma^{\nu\rho} = \delta_\mu^\nu - u^\rho t_\mu = \gamma_\mu^\nu. \quad (108)$$

The comoving energy current $U_{\text{int}}^{c\mu}$ (which would be the same as the ordinary internal energy current U_{int}^μ in a locally comoving frame – meaning a Galilean gauge with $e^\mu = u^\mu$ at the position under consideration) will be expressible in the form

$$U_{\text{int}}^{c\mu} = u^\mu U_{\text{int}} - \frac{\partial \Lambda_{\text{int}}}{\partial n_f} n_\perp^\mu, \quad (109)$$

in terms of the ordinary internal energy $U_{\text{int}} = -t_\mu T_{\text{int}\nu}^\mu e^\nu$ which can be seen to be given simply by

$$U_{\text{int}} = \frac{\partial \Lambda_{\text{int}}}{\partial n^A} n^A - \Lambda_{\text{int}}. \quad (110)$$

Note that the first term on the right of this formula is not something that depends on the elastic solidity, but something that – except in the absence of the entrainment effect [16] discussed in Subsection 4.4 – will still be present even in the fluid limit. This means that (contrary to what has been suggested in the literature [8]) in the presence of entrainment it will never be permissible to simply identify the internal action density with the negative of the internal energy density.

As in the fluid case, we can go on to construct locally defined material 4-momenta

$$\mu_\nu^f = p_\nu^f + \chi_\nu^f, \quad \mu_\nu^c = p_\nu^c + \chi_\nu^c \quad (111)$$

from which, after allowance for the non local effect of gravity, we finally get the total momenta for the free and confined particles,

$$\pi_\nu^f = \mu_\nu^f - m \phi t_\nu, \quad \pi_\nu^c = \mu_\nu^c - m \phi t_\nu, \quad (112)$$

in terms of which the complete stress energy tensor will be expressible as

$$T_\nu^\mu = -\Theta s^\mu t_\nu + \pi_\nu^f n_f^\mu + \pi_\nu^c n_c^\mu + \Psi \delta_\nu^\mu - \mathcal{P}_\nu^\mu. \quad (113)$$

with

$$\mathcal{P}_\nu^\mu = S_\nu^\mu + \Lambda_{\text{int}} \gamma_\nu^\mu + n_\perp^\mu \frac{\partial \Lambda_{\text{int}}}{\partial n_\perp^\nu}. \quad (114)$$

4.3 Elastic energy contribution

As in the multiconstituent fluid case [16] it is useful to decompose the internal action function in the form

$$\Lambda_{\text{int}} = \Lambda_{\text{ins}} + \Lambda_{\text{ent}} , \quad (115)$$

so as to obtain a corresponding decomposition

$$S_{\mu}^{\nu} = S_{\text{ins } \mu}^{\nu} + S_{\text{ent } \mu}^{\nu} , \quad (116)$$

in which Λ_{ent} and $S_{\text{ent } \mu}^{\nu}$ are the parts attributable to entrainment, and Λ_{ins} and $S_{\text{ins } \mu}^{\nu}$ are the static internal contributions that remain when the relative current contributions n^A are set to zero. For this static part (but not for the rest) the action density will just be the opposite of the elastic energy density ϵ as defined by setting

$$U_{\text{ins}} = \epsilon , \quad (117)$$

which can be seen from (110) to correspond to

$$\Lambda_{\text{ins}} = -\epsilon . \quad (118)$$

It will commonly be convenient to further decompose this static energy contribution in the form

$$\epsilon_{\text{ins}} = \epsilon_{\circ} + \epsilon_{\text{sol}} , \quad \epsilon_{\circ} = -\Lambda_{\circ} , \quad (119)$$

in which ϵ_{\circ} is the part that remains in a relaxed configuration for which ϵ is minimised for given values of the independent current components n_c , n_f , n^A , and of the determinant of the induced metric γ^{AB} . Fixing this determinant is equivalent to fixing the value of the conserved number density n_I that is specified by (57). We shall use the notation $\check{\gamma}^{AB}$ and $\check{\gamma}_{AB}$ respectively for the corresponding relaxed values of γ^{AB} and its inverse γ_{AB} (as defined by $\gamma_{AB}\gamma^{BC} = \delta_A^C$) at which, for given values of n_I , n_c , n_f , the maximisation occurs. Thus substitution of $\check{\gamma}^{AB}$ for γ^{AB} in the solidity term ϵ_{sol} or in the total ϵ or will give a generically reduced value

$$\check{\epsilon}_{\text{sol}} \leq \epsilon_{\text{sol}} , \quad \check{\epsilon} \leq \epsilon , \quad (120)$$

but it will have no effect on the relaxed part, or on the ionic number density n_I , for which we simply get

$$\check{\epsilon}_{\circ} = \epsilon_{\circ} , \quad \check{n}_I = n_I . \quad (121)$$

The relaxed contribution will evidently be of ordinary (albeit non-barotropic) perfect fluid type with a generic variation of the form

$$\delta \epsilon_{\circ} = -\delta \Lambda_{\circ} = \Theta_{\circ} \delta s + \chi_{\circ}^f \delta n_f + \chi_{\circ}^c \delta n_c + \chi_{\circ}^s \delta n_I - \lambda_{\circ A}^s \delta q^A , \quad (122)$$

(in which the final term allows for the possibility of built in inhomogeneity in addition to the stratification due just to the variation of the atomic number ratio A_c) with

$$\delta n_I = \frac{1}{2} n_I \gamma_{AB} \delta \gamma^{AB} + \frac{\partial n_I}{\partial q^A} \delta q_A. \quad (123)$$

According to (95) the relaxed contribution to the pressure tensor (101) will be given by

$$P_{\circ}^{AB} = (\Theta_{\circ} s + \chi_{\circ}^f n_f + \chi_{\circ}^c n_c) \gamma^{AB} - S_{\circ}^{AB}, \quad S_{\circ}^{AB} = (\epsilon_{\circ} - \chi_{\circ}^s n_I) \gamma^{AB}, \quad (124)$$

so one finally obtains an expression of the familiar isotropic form

$$P_{\circ}^{AB} = P_{\circ} \gamma^{AB}, \quad P_{\circ} = \chi_{\circ}^f n_f + \chi_{\circ}^I n_I - \epsilon_{\circ}, \quad \chi_{\circ}^I = \chi_{\circ}^s + A_c \chi_{\circ}^c + \frac{s}{n_I} \Theta_{\circ}. \quad (125)$$

Let us now consider the solidity contribution ϵ_{sol} whose job is to allow for the effect of deviations of γ_{AB} from its relaxed value $\check{\gamma}_{AB}$ (as determined by the scalars n^f , n_c , n_I). Such deviations can conveniently be accounted for [2] in terms of the constant volume shear tensor whose material base space representation is specified as

$$\varsigma_{AB} = \frac{1}{2} (\gamma_{AB} - \check{\gamma}_{AB}), \quad (126)$$

which means that the corresponding space time tensor will be given by

$$\varsigma_{\mu\nu} = \varsigma_{AB} q_{,\mu}^A q_{,\nu}^B = \frac{1}{2} (\gamma_{\mu\nu} - \check{\gamma}_{\mu\nu}). \quad (127)$$

In most applications to behaviour of a perfectly elastic (rather than plastic or other more complicated) kind, it will be sufficient to use an ansatz of quasi-Hookean type [2], meaning one in which the solidity contribution has a homogeneously quadratic dependence on the deviation (126) in the sense that it will be given by an expression of the form

$$\epsilon_{\text{sol}} = \frac{1}{2} \check{\Sigma}^{ABCD} \varsigma_{AB} \varsigma_{CD}, \quad (128)$$

with

$$\check{\Sigma}^{ABCD} = \check{\Sigma}^{(AB)(CD)} = \check{\Sigma}^{CDAB}, \quad (129)$$

in which $\check{\Sigma}^{ABCD}$ is the relevant shear elasticity tensor, for which the check symbol is used to indicate that, for a given value of the material position coordinates q^A , it depends only on the scalars s , n_f , n_c and (via n_I) on the relaxed metric $\check{\gamma}_{AB}$. The condition that γ_{AB} and $\check{\gamma}_{AB}$ must have the same determinant entails that on the 3 dimensional material base the symmetric shear tensor ς_{AB} will have only 5 (instead of 6) independent components, and more specifically that to first order it will be trace free with respect to either the actual metric γ_{AB} or the relaxed $\check{\gamma}_{AB}$. It is therefore necessary to impose a corresponding restriction to completely fix the specification of the solidity tensor $\check{\Sigma}^{ABCD}$,

which can most conveniently be done [2] by requiring that it be trace free with respect to the relaxed metric

$$\check{\Sigma}^{\text{ABCD}} \check{\gamma}_{\text{CD}} = 0. \quad (130)$$

The specification of the solidity (i.e. shear elasticity) tensor is not by itself sufficient to complete the specification of the elastic system, as it is also necessary to specify the dependence on n_{I} of the relaxed inverse metric $\check{\gamma}^{\text{AB}}$. The simplest possibility is that of what has been termed a perfect solid [2], meaning one at which the elastic structure at each material position is isotropic with respect to the relaxed metric, which in that case can vary only by a conformal factor. This means that it will be given in terms of its value γ_0^{AB} say at some fixed reference value n_0 say of the ionic number density n_{I} by

$$\check{\gamma}^{\text{AB}} = (n_{\text{I}}/n_0)^{2/3} \gamma_0^{\text{AB}}, \quad \check{\gamma}_{\text{AB}} = (n_0/n_{\text{I}})^{2/3} \gamma_{0\text{AB}}. \quad (131)$$

In the case of a solid structure that is isotropic (as will typically be the case on a macroscopic scale after averaging over randomly oriented mesoscopic crystals) the rigidity tensor in the quasi Hookean ansatz will simply have to be given in terms of the relevant scalar shear modulus $\check{\Sigma}$ by the formula

$$\check{\Sigma}^{\text{ABCD}} = 2\check{\Sigma} (\check{\gamma}^{\text{A(C}} \check{\gamma}^{\text{D)B}} - \frac{1}{3} \check{\gamma}^{\text{AB}} \check{\gamma}^{\text{CD}}), \quad (132)$$

where the scalar $\check{\Sigma}$ is the rigidity modulus which is usually denoted by μ in the literature, a symbol which is already being used in the general formalism used here to design the material momentum components.

Equation (128) will then give the simple formula

$$\Lambda_{\text{sol}} = -\epsilon_{\text{sol}} = -\check{\Sigma} \varsigma^2, \quad (133)$$

in which the scalar shear magnitude ς is defined by the formula

$$\varsigma^2 = \check{\gamma}^{\text{AB}} \gamma^{\text{CD}} \varsigma_{\text{BC}} \varsigma_{\text{DA}} - \frac{1}{3} (\check{\gamma}^{\text{AB}} \varsigma_{\text{AB}})^2, \quad (134)$$

in which the final term will in practice be negligible since of quartic order, $\mathcal{O}\{\varsigma^4\}$, in the small ς limit that is relevant, as the trace is already of quadratic order, $\check{\gamma}^{\text{AB}} \varsigma_{\text{AB}} = \mathcal{O}\{\varsigma^2\}$. Under these conditions the solidity contribution will provide a pressure tensor given by the formula

$$P_{\text{sol}}^{\text{AB}} = -\check{\Sigma}^{\text{ABCD}} \varsigma_{\text{AB}} + P_{\text{sol}} \gamma^{\text{AB}}, \quad (135)$$

of which the final term is a pressure contribution given by

$$P_{\text{sol}} = \left(s \frac{\partial \Sigma}{\partial s} + n_{\text{f}} \frac{\partial \Sigma}{\partial n_{\text{f}}} + n_{\text{c}} \frac{\partial \Sigma}{\partial n_{\text{c}}} + n_{\text{I}} \frac{\partial \Sigma}{\partial n_{\text{I}}} + \frac{\Sigma}{3} \right) \varsigma^2, \quad (136)$$

which, since it is of quadratic order in ς , will be negligible compared with the first (linear order) term for most practical purposes. (It is to be noted that in the final term

of (136) the sign given here corrects an error in the sign of the corresponding term in the relevant equation (6.19) as written in the original treatment [2] of the perfect solid model.) The corresponding pressure adjustment contribution for the canonical formula (97) will be given by

$$\mathcal{P}^\mu_\nu = \gamma_{\nu\lambda} \tilde{\Sigma}^{\lambda\mu\rho\sigma} \varsigma_{\rho\sigma} - \left(n_{\text{I}} \frac{\partial \Sigma}{\partial n_{\text{I}}} + \frac{4\Sigma}{3} \right) \varsigma^2 \gamma^\mu_\nu, \quad (137)$$

in which, again, the quadratic order term at the end will be negligible in practice.

4.4 Entrainment contribution

In general the entrainment action function Λ_{ent} will depend, for a given values of q^A , on the relative current components $n^A = q^\perp_{,\nu} n^\nu_\perp$ as well as on the scalar magnitudes n_{f} , n_{c} , and the induced metric components γ^{AB} , so its generic variation will have the form

$$\delta \Lambda_{\text{ent}} = -\Theta_{\text{ent}} \delta s - \chi_{\text{ent}}^{\text{f}} \delta n_{\text{f}} - \chi_{\text{ent}}^{\text{c}} \delta n_{\text{c}} + \frac{\partial \Lambda_{\text{ent}}}{\partial \gamma_{AB}} \delta \gamma^{AB} + \frac{\partial \Lambda_{\text{ent}}}{\partial n^A} \delta n^A + \frac{\partial \Lambda_{\text{ent}}}{\partial q^A} \delta q^A. \quad (138)$$

The entrainment action Λ_{ent} is characterised by the condition that it vanishes when the relative current components n^A are set to zero, so when these components are sufficiently small, as will typically be the case, it will be a good approximation to take this contribution to have the homogeneous quadratic form

$$\Lambda_{\text{ent}} = \frac{1}{2n_{\text{f}}} m_{\text{AB}}^\perp n^A n^B, \quad (139)$$

in which the entrainment mass tensor has components m_{AB}^\perp , that (like the static action contribution) are independent of the current components n^A , so that the corresponding partial derivative in (138) will be given by

$$\frac{\partial \Lambda_{\text{ent}}}{\partial n^A} = \frac{\partial \Lambda_{\text{ent}}}{\partial n^A} = \frac{1}{n_{\text{f}}} m_{\text{AB}}^\perp n^B. \quad (140)$$

It is conceivable that the relaxed action function might involve a built in anisotropy favoring relative currents in some particular direction, but in cases of the simplest kind, to which the remainder of this subsection and the next will be restricted, this function Λ_{ent} will be of purely fluid type in the sense that for given values of q^A it will depend only on the set of five scalar magnitudes consisting of n_{I} , n_{c} , n_{f} , s , together with the relative current magnitude n_\perp that is defined in terms of the (unrelaxed) metric value γ_{AB} which – using the material index lowering operation specified by the induced metric γ^{AB} – will be given by

$$n_\perp^2 = n^A n_A, \quad n_A = \gamma_{AB} n^B, \quad \gamma_{AB} \gamma^{BC} = \delta_A^C. \quad (141)$$

Such a functional dependence provides an expansion

$$\delta\Lambda_{\text{ent}} = -\Theta_{\text{ent}}\delta s - \chi_{\text{ent}}^{\text{f}}\delta n_{\text{f}} - \chi_{\text{ent}}^{\text{c}}\delta n_{\text{c}} - \chi_{\text{ent}}^{\text{S}}\delta n_{\text{I}} + \lambda_{\text{ent A}}^{\text{S}}\delta q^{\text{A}} + \frac{\partial\Lambda_{\text{ent}}}{\partial n_{\perp}^2}\delta n_{\perp}^2, \quad (142)$$

of similar form to the perfect fluid contribution (122) but with an extra term involving a partial derivative that provides an expression of the form (140) in terms an isotropic mass tensor given by

$$m_{\text{AB}}^{\perp} = m_{\text{c}}^{\text{f}}\gamma_{\text{AB}}, \quad m_{\text{c}}^{\text{f}} = 2n_{\text{f}}\frac{\partial\Lambda_{\text{ent}}}{\partial n_{\perp}^2}, \quad (143)$$

while the other partial derivatives in (138) will be given by

$$\frac{\partial\Lambda_{\text{ent}}}{\partial\gamma_{\text{AB}}} = -\frac{1}{2}\left(\frac{m_{\text{c}}^{\text{f}}}{n_{\text{f}}}n_{\text{A}}n_{\text{B}} + \chi_{\text{ent}}^{\text{S}}n_{\text{I}}\gamma_{\text{AB}}\right), \quad \frac{\partial\Lambda_{\text{ent}}}{\partial q^{\text{A}}} = \lambda_{\text{ent A}}^{\text{S}} - \chi_{\text{ent}}^{\text{S}}\frac{\partial n_{\text{I}}}{\partial q^{\text{A}}}. \quad (144)$$

The scalar m_{c}^{f} introduced in this way is identifiable as the increment $m_{\text{c}}^{\text{f}} = m_{\star} - m$ of the effective mass m_{\star} of the free baryons (meaning the superfluid neutrons) as compared with the ordinary baryonic mass m . This mass increment is expected to be positive (and in some layers large) [16] in the solid neutron star crust, but (moderately) negative in the fluid layers below.

4.5 Relaxed action contribution

It will be useful for many purposes to replace the decomposition (115) of the internal action density by an alternative decomposition of the form

$$\Lambda_{\text{int}} = \Lambda_{\text{lax}} + \Lambda_{\text{sol}}, \quad (145)$$

in which the relaxed – meaning shear independent – part will evidently consist of the combination

$$\Lambda_{\text{lax}} = \Lambda_{\text{O}} + \Lambda_{\text{ent}}. \quad (146)$$

The use of such a combination is particularly convenient whenever the entrainment contribution is of the isotropic type characterised by the variation expansion (142), in which case the complete relaxed action density will have a variation given by an expansion of the analogous form

$$\delta\Lambda_{\text{lax}} = -\Theta_{\text{lax}}\delta s - \chi_{\text{lax}}^{\text{f}}\delta n_{\text{f}} - \chi_{\text{lax}}^{\text{c}}\delta n_{\text{c}} - \chi_{\text{ent}}^{\text{S}}\delta n_{\text{I}} + \lambda_{\text{lax A}}^{\text{S}}\delta q^{\text{A}} + \frac{m_{\text{c}}^{\text{f}}}{2n_{\text{f}}}\delta n_{\perp}^2, \quad (147)$$

with

$$\begin{aligned} \Theta_{\text{lax}} &= \Theta_{\text{O}} + \Theta_{\text{ent}}^{\text{f}}, & \chi_{\text{lax}}^{\text{f}} &= \chi_{\text{O}}^{\text{f}} + \chi_{\text{ent}}^{\text{f}}, & \chi_{\text{lax}}^{\text{c}} &= \chi_{\text{O}}^{\text{c}} + \chi_{\text{ent}}^{\text{c}}, \\ \chi_{\text{lax}}^{\text{S}} &= \chi_{\text{O}}^{\text{S}} + \chi_{\text{ent}}^{\text{S}}, & \lambda_{\text{lax A}}^{\text{S}} &= \lambda_{\text{O A}}^{\text{S}} + \lambda_{\text{ent A}}^{\text{S}}. \end{aligned} \quad (148)$$

It can be seen that the relaxed contribution to the “live” action variation (93) will simplify to provide an expression of the form

$$\delta^\heartsuit \Lambda_{\text{lax}} = -\Theta_{\text{lax}} \delta s + \chi_{\text{lax}\nu}^f \delta n_f^\nu + \chi_{\text{lax}\nu}^c \delta n_c^\nu - \chi^S \delta^\heartsuit n_I + \lambda_{\text{lax}\Lambda}^S \delta q^A, \quad (149)$$

in which $\chi_{\text{lax}\nu}^f$ and $\chi_{\text{lax}\nu}^c$ can be read out as the relaxed parts of the internal momenta given by (95). Since we can write $\delta^\heartsuit n_I = t_\mu \delta^\heartsuit n_I^\mu$, it can be seen that, as the analogue of these free and confined particle 4-momentum contributions, we shall also be able to read out a corresponding ionic stratification 4-momentum contribution $\chi_{\text{lax}\nu}^S$, so as to obtain a complete set of relaxed internal momentum covectors that will be given by

$$\chi_{\text{lax}\mu}^f = -\chi_{\text{lax}}^f t_\mu + \frac{m_c^f}{n_f} \gamma_{\mu\nu} n_\perp^\nu, \quad \chi_{\text{lax}\mu}^c = -\chi_{\text{lax}}^c t_\mu - \frac{m_c^f}{n_c} \gamma_{\mu\nu} n_\perp^\nu, \quad \chi_{\text{lax}\nu}^S = -\chi_{\text{lax}}^S t_\nu. \quad (150)$$

In the formula (94) for the extra stress, it transpires that the contributions involving the mass increment m_c^f (proportional to $\partial \Lambda_{\text{lax}} / \partial n_\perp^2$) will cancel out, leaving the expression

$$\mathcal{P}_{\text{lax}\Lambda}^\nu = -\chi_{\text{lax}}^S n_I \gamma_{\Lambda\mu} q_{,\mu}^B \gamma^{\mu\nu}, \quad (151)$$

in which the part due to stratification is all that remains. It can be seen that the corresponding space time tensor will be given simply by

$$\mathcal{P}_{\text{lax}\mu}^\nu = \chi_{\text{lax}}^S n_I (u^\nu t_\mu - \delta_\mu^\nu). \quad (152)$$

In terms of their contributions to the total momenta, which will simply be given by

$$\pi_{\text{lax}\nu}^\emptyset = -\Theta_{\text{lax}} t_\nu, \quad \pi_{\text{lax}\nu}^f = \chi_{\text{lax}\nu}^f, \quad \pi_{\text{lax}\nu}^c = \chi_{\text{lax}\nu}^c, \quad \pi_{\text{lax}\nu}^S = \chi_{\text{lax}\nu}^S, \quad (153)$$

the corresponding elastically relaxed force contributions acting on the entropy current and on the free and confined particle currents will be given by an ansatz of the standard form (39) which gives

$$f_{\text{lax}\nu}^\emptyset = 2s^\mu \nabla_{[\mu} \pi_{\text{lax}\nu]}^\emptyset + \pi_{\text{lax}\nu}^\emptyset \nabla_\mu s^\mu, \quad (154)$$

$$f_{\text{lax}\nu}^f = 2n_f^\mu \nabla_{[\mu} \pi_{\text{lax}\nu]}^f + \pi_{\text{lax}\nu}^f \nabla_\mu n_f^\mu, \quad (155)$$

and

$$f_{\text{lax}\nu}^c = 2n_c^\mu \nabla_{[\mu} \pi_{\text{lax}\nu]}^c + \pi_{\text{lax}\nu}^c \nabla_\mu n_c^\mu, \quad (156)$$

while for the analogously defined force contribution (40) due to stratification acting on the underlying ionic lattice, it can be seen that similar reasoning leads to an expression of the slightly different form

$$f_{\text{lax}\nu}^S = 2n_I^\mu \nabla_{[\mu} \pi_{\text{lax}\nu]}^S + \lambda_{\text{lax}\nu}^S, \quad \lambda_{\text{lax}\nu}^S = \lambda_{\text{lax}\Lambda}^S q_{,\nu}^A. \quad (157)$$

In terms of the amalgamated ionic 4-momentum contribution defined by

$$\pi_{\text{lax}\nu}^{\text{I}} = \pi_{\text{lax}\nu}^{\text{S}} + A_{\text{c}} \pi_{\text{lax}\nu}^{\text{c}} + (s/n_{\text{I}}) \pi_{\text{lax}\nu}^{\emptyset}, \quad (158)$$

the corresponding contribution

$$f_{\text{lax}\nu}^{\text{I}} = f_{\text{lax}\nu}^{\text{S}} + f_{\text{lax}\nu}^{\text{c}} + f_{\text{lax}\nu}^{\emptyset}, \quad (159)$$

to the amalgamated ionic force density (19) can be seen to be expressible directly by the formula

$$f_{\text{lax}\nu}^{\text{I}} = 2n_{\text{I}}^{\mu} \nabla_{[\mu} \pi_{\text{lax}\nu]}^{\text{I}} + n_{\text{I}}^{\mu} \pi_{\text{lax}\mu}^{\text{c}} \nabla_{\nu} A_{\text{c}} - n_{\text{I}} \Theta_{\text{lax}} \nabla_{\nu} (s/n_{\text{I}}) + \lambda_{\text{lax}\nu}^{\text{S}}, \quad (160)$$

while the associated contribution to the stress energy tensor (36) will be given neatly by

$$T_{\text{lax}\nu}^{\mu} = (\Lambda_{\text{lax}} - n_{\text{f}}^{\rho} \pi_{\text{lax}\rho}^{\text{f}} - n_{\text{I}}^{\rho} \pi_{\text{lax}\rho}^{\text{I}}) \delta_{\nu}^{\mu} + n_{\text{f}}^{\mu} \pi_{\text{lax}\nu}^{\text{f}} + n_{\text{I}}^{\mu} \pi_{\text{lax}\nu}^{\text{I}}. \quad (161)$$

4.6 Complete description for perfect conducting solid.

Replacing the relaxed contribution (158) by the corresponding total ionic 4-momentum covector given (in terms of the confined atomic number A_{c}) by

$$\pi_{\nu}^{\text{I}} = \pi_{\nu}^{\text{S}} + A_{\text{c}} \pi_{\nu}^{\text{c}} - (s/n_{\text{I}}) \Theta t_{\nu}, \quad (162)$$

we can apply an analogous tidying up operation to the complete stress energy tensor (36) which will thereby acquire the form

$$T_{\nu}^{\mu} = (\Lambda - n_{\text{f}}^{\rho} \pi_{\rho}^{\text{f}} - n_{\text{I}}^{\rho} \pi_{\rho}^{\text{I}}) \delta_{\nu}^{\mu} + n_{\text{f}}^{\mu} \pi_{\nu}^{\text{f}} + n_{\text{I}}^{\mu} \pi_{\nu}^{\text{I}} - \mathcal{P}_{\text{sol}\nu}^{\mu}, \quad (163)$$

in which the only manifest allowance for the effects of solid rigidity is in the final term. In the case of a perfect conducting solid – meaning one whose structure is fully isotropic with respect to the relaxed metric – this final term $\mathcal{P}_{\text{sol}\nu}^{\mu}$ will be given by the formula (137) which will be expressible to first order in the shear amplitude ς by a prescription of the simple form

$$\mathcal{P}_{\text{sol}\nu}^{\mu} = 2\Sigma \gamma^{\mu\rho} \varsigma_{\rho\nu} + \mathcal{O}\{\varsigma^2\}, \quad (164)$$

in which Σ is the ordinary shear modulus (which elsewhere is commonly denoted by the symbol μ that, in the present context, has already been used for the designation of the material momentum components). In principle, the other terms in (163) will also be influenced by the solid rigidity contribution Λ_{sol} , but since such effects too will be of quadratic order in ς , they will be effectively negligible in applications of the usual kind, in which deviations from an elastically relaxed configuration are small. This means that for practical purposes, in a conducting solid of the perfect (meaning intrinsically isotropic) type, the deviation from behaviour of (multiconstituent) fluid type [13] will be entirely contained in the extra term given by (164).

5 Non conservative generalisation

5.1 Dissipative interpolation

In the non-dissipative models described above, it can be seen that the introduction of the entropy density s as an independent variable was in practice redundant, since its effects could be allowed for simply by a readjustment of the stratification, because the ratio s/n_I was fixed on each material world line so that it depended only on the material base space variables q^A .

The reason for taking the trouble of introducing it is that the entropy density will acquire a non-trivial role as soon as these convectively conducting solid models are generalised to allow for dissipative effects in the manner that has recently been described in detail [16] for the fluid case. The second law of thermodynamics tells us that, in a dissipative application, the entropy current s^ν need not satisfy the conservation condition (73), but that – for a system that is closed in the sense of being thermodynamically isolated from the rest of the universe – the model must be such as to ensure satisfaction of the inequality

$$\nabla_\nu s^\nu \geq 0. \quad (165)$$

On the assumption that it is isolated not just thermodynamically but in the stronger mechanical sense of (46), we have seen that a system characterised by an action of the kind presented above, and thus by a stress energy tensor of the form (56), must automatically satisfy the energy conservation identity (52). Subject to the usual presumption that the temperature Θ is positive, this means that the second law requirement (165) will be expressible for such a model as

$$(\mathcal{E}^c - \mathcal{E}^f) \nabla_\nu n_f^\nu + n_f^\mu \varpi_{\mu\nu}^f u^\nu \geq 0. \quad (166)$$

The simplest and most obviously natural way of satisfying this positivity requirement starts by postulating that the two terms in (166) are separately positive. For the first term this leads to ansatz to the effect that relevant neutron fluid creation rate should be given by an expression of the form

$$\nabla_\nu n_f^\nu = \kappa (\mathcal{E}^c - \mathcal{E}^f). \quad (167)$$

in terms of some positive transfusion coefficient κ that might be expected to be a sensitive function of the temperature, Θ , on which the weak interactions that would be involved are known [18] to be highly dependent. (It is to be noticed that the formula (167) will be preserved by the chemical gauge transformation (62) for any fixed value of the adjustment parameter ϵ .)

The treatment of the second term in (166) is not so simple, because it is necessary to respect the degeneracy requirement (69) that ensures that the mesoscopically averaged vorticity $\varpi_{\mu\nu}^f$ is orthogonal to two-surfaces that represent the flux of quantised vorticity tubes with generators of the form $u_f^\nu + V_f^\nu$ for some spacelike vector V_f^ν such that

$$\varpi_{\mu\nu}^f (u_f^\nu + V_f^\nu) = 0, \quad V_f^\nu t_\nu = 0. \quad (168)$$

In terms of such a vector, the positivity condition on the second term in (166) will be expressable (assuming positivity of n_f) as

$$u^\mu \varpi_{\mu\nu}^f V_f^\nu \geq 0, \quad (169)$$

so a vector of the required form will be given by an expression of the form

$$V_f^\nu = -\frac{c_r}{w^f} \gamma^{\nu\rho} \varpi_{\rho\sigma}^f u^\sigma, \quad (170)$$

in which c_r is a positive drag coefficient that has been adjusted so as to be dimensionless by the inclusion of the denominator w^f , which is defined as the magnitude of the (spacelike) vorticity vector $w^{f\mu} = \frac{1}{2} \varepsilon^{\mu\nu\rho} \varpi_{\nu\rho}^f$. This provides a superfluid equation of motion of the form

$$u_f^\rho \varpi_{\rho\sigma}^f \gamma^{\sigma\mu} = c_r w^f v_r^\mu, \quad (171)$$

in which v_r^μ is a relative flow velocity, namely that of the medium relative to the vortex sheets, as defined [15] in terms of the relevant orthogonal projection operator by

$$v_r^\mu = \perp_\nu^\mu u^\nu, \quad \perp_\nu^\mu = (w^f)^{-2} \gamma^{\mu\rho} \gamma^{\sigma\tau} \varpi_{\nu\sigma}^f \varpi_{\rho\tau}^f. \quad (172)$$

On the basis of work by Jones [19], a formula giving a rather low value for the required drag coefficient c_r , in the low temperature limit, as a function of the relevant densities, n_I , n_c , n_f , has been provided by Langlois *et al.* [6], but it depends on microscopic parameters that are difficult to evaluate, and its validity is in any case a subject of controversy. In the very different picture developped by Alpar *et al.* [20] it has been suggested that the result will be highly temperature dependent, and that the effect will be describable as (nonlinear) “creep” rather than ordinary (linear) drag, in the sense that the appropriate coefficient c_r will not just depend on the relevant scalar densities as well as the temperature Θ , but that it will also be strongly dependent on the magnitude v_r of the relative velocity (172) (according to a formula of the form $c_r \propto v_r^{-1} \text{arcsinh}\{v_r/\bar{v}_r\}$ for some velocity independent – but temperature sensitive – quantity \bar{v}_r).

It is to be remarked that the chemical reaction rate formula (167) provides an interpolation between the non dissipative limits $\kappa \rightarrow \infty$, namely the thermal equilibrium case, and the variational case $\kappa \rightarrow 0$, for which free and confined particle currents are separately conserved. Similarly the drag or creep formula (171) provides an interpolation between the non dissipative limits $c_r \rightarrow \infty$, namely the perfect pinning case, and variational case $c_r \rightarrow 0$, for which the vortices are freely transported by the superfluid flow.

5.2 Conclusion

Subject to the prescription of two suitable secondary equations of state for the coefficients κ and c_r as functions of the relevant variables, the foregoing equations (167) and (171) constitute a complete system of equations of motion for the nine independent component variables (which can be considered to be the four superfluid current components n^ν , and the five materially convected components q^A , n_c and s) when used in conjunction with the baryon conservation condition (51) and the condition (46) of conservation of the stress energy tensor T^μ_ν that is obtained, according to (163), from the master function Λ , whose prescription, by a suitable primary equation of state is described in Section 4.

The stress energy conservation condition (46) expresses the requirement that the system should be effectively isolated, not just mechanically, but also thermally. This last requirement will not be entirely realistic when the transfusive adjustments governed by (167) are taking place [18], since the beta processes involved will create neutrinos for which the stellar medium will be effectively transparent so that instead of being locally confined they will rapidly escape from the system. The ensuing heat loss can easily be formally taken into account [15] by replacing (46) by a generalisation in which there is an extra term so, that it takes the form

$$\nabla_\mu T^\mu_\nu = -\rho \nabla_\nu \phi + Q t_\mu, \quad (173)$$

in which Q represents the heat loss rate per unit volume. This means that the energy conservation identity (52) will need to be replaced by the energy loss formula

$$Q + \Theta \nabla_\nu s^\nu = (\mathcal{E}^c - \mathcal{E}^f) \nabla_\nu n_f^\nu + n_f^\nu \varpi_{\mu\nu}^f u^\nu = \kappa (\mathcal{E}^c - \mathcal{E}^f)^2 + c_r n_f w^f v_r^2, \quad (174)$$

in which the right hand side is the combination of terms that (by the non negativity of the coefficients κ and c_r) has been made to satisfy the positivity condition (166).

If Q is prescribed (by what would be a third secondary equation of state) as a function of the relevant densities – particularly that of the entropy which determines the temperature Θ – then (174) will in principle provide what is needed for calculating the evolution of the entropy current. However, as the temperature dependence of Q is likely to be highly sensitive [18], such an approach might not be accurate in practice. It might be more realistic to suppose that the temperature would adjust itself to the rather low roughly constant value needed to avoid accumulation of entropy, which incidentally is the consideration that justifies, as a reasonable approximation, the neglect of heat conduction in this work. This means adopting the conservation postulate (73) to the effect that the second term on the left of (174) will simply drop out. The implication is that Q will be given just by the positive terms on the right of (174), and that it will therefore vanish in the non dissipative limit cases that are obtainable, as described in Subsection 3.3, by taking infinite or zero values of the dissipation coefficients κ and c_r .

Acknowledgements. The authors wish to thank S. Bonazzola, N. Chamel, D. Langlois, P. Haensel and L. Samuelsson for helpful discussions on various occasions.

References

- [1] J.E.Horvath, “Pulsar astrophysics: the glitch phenomenon”, *Int. J. Mod. Phys.* **D13** (2004) 1327-1334. [astro-ph/0404324]
- [2] B. Carter, H. Quintana, “Foundations of general relativistic high-pressure elasticity theory”, *Proc. Roy. Soc. Lond.* **A331** (1972) 57-83.
- [3] B. Carter, H. Quintana, “Stationary Elastic Rotational Deformation of a Relativistic Neutron Star Model”, *Astrophys. J.* **202** (1975) 511-522.
- [4] G. Mendell, “Superfluid hydrodynamics in rotating neutron stars. I Nondissipative equations”, *Astrophys. J.* **380** (1991) 515-529.
- [5] B. Carter, I.M. Khalatnikov, “Equivalence of convective and potential variational derivations of covariant superfluid dynamics”, *Phys. Rev.* **D45** (1992) 4536-4544.
- [6] D. Langlois, D. Sedrakian, B. Carter, “Differential rotation of relativistic superfluid in neutron stars”, *Mon. Not. R. Astr. Soc.*, **297**, pp 1198-1201 (1998). [astro-ph/9711 042]
- [7] G.L. Comer, R. Joint, “A relativistic mean field model for entrainment in general relativity”, *Phys. Rev.* **D68** (2003) 023002. [gr-qc/0212083]
- [8] R. Prix, G.L. Comer, N. Andersson, “Slowly rotating superfluid neutron star model with entrainment”, *Astron Astrophys.* **381** (2002) 178-196. [astro-ph/0107176]
- [9] R. Prix, J. Novak, G.L. Comer, “Relativistic numerical models for stationary superfluid neutron stars” *Phys. Rev.* **D71** (2005) 043005. [gr-qc/0410023]
- [10] N. Chamel, B. Carter, “Effect of entrainment on stress on stress and pulsar glitches in stratified neutron star crust”. [astro-ph/0503044 v2].
- [11] B. Carter, E. Chachoua, N. Chamel, “Covariant Newtonian and Relativistic dynamics of (magneto)-elastic solid model for neutron star crust.” (preprint, ObsPM + CAMK, Dec., 2004) [gr-qc/0507006]
- [12] B. Carter, N. Chamel, “Covariant analysis of Newtonian multi-fluid models for neutron stars: I Milne - Cartan structure and variational formulation”, *Int J. Mod. Phys.* **D13** (2004) 291-325. [astro-ph/0305186]
- [13] B. Carter, N. Chamel, “Covariant analysis of Newtonian multi-fluid models for neutron stars: II Stress - energy tensors and virial theorems”, *Int J. Mod. Phys.* **D14** (2005) 717-748. [astro-ph/0312414]

- [14] B. Carter, L. Samuelsson, “Relativistic mechanics of neutron superfluid in (magneto) elastic star crust.” [Meudon and Southampton, preprint, Dec., 2005]
- [15] B. Carter, N. Chamel, “Covariant analysis of Newtonian multi-fluid models for neutron stars: III Transvective, viscous and superfluid drag dissipation”, *Int J. Mod. Phys. D* **14** (2005) 7149-774. [astro-ph/0410660]/]
- [16] B. Carter, N. Chamel, P. Haensel, “Entrainment coefficient and effective mass for conduction neutrons in neutron star crust: 2 macroscopic treatment”, B. Carter, N. Chamel, P. Haensel. [astro-ph/0408083 v3]
- [17] B. Carter, “Rheometric Structure Theory, Convective Differentiation and Continuum Electrodynamics”, *Proc. Roy. Soc. Lond.* **A372** (1980) 169-200.
- [18] P. Haensel, “Non-equilibrium neutrino emissivities and opacities of neutron star matter”, *Astron. Astroph.* (262) (1992) 131-137.
- [19] P.B. Jones, “Rotation of the neutron-drip superfluid in neutron stars: period discontinuities and internal temperatures”, *Mon. Not. R. Astr. Soc.*, **246** (1990) 315-323.
- [20] A. Alpar, K. Cheng, D. Pines, “Vortex creep and the internal temperature of neutron stars: linear and non-linear response to a glitch” *Astroph. J* **346** (1989) 823-832.